New derivation operators in dual Finsler geometry

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Abstract
In this paper work I will introduce the dynamical covariant derivative operator on the dual space of a given Finsler space using the idea of Legendre duality between the lagrangian formalism and the hamiltonian formalism. Making use of this new operator introduced I will proof that the solution curve equations and their deviations have an elegant geometrical expression.

Keywords: Tangent Bundle, Cotangent Bundle, Lagrangian, Hamiltonian, Legendre Duality, Geodesic Spray, Ehresmann Connection, Berwald connection, geodesic, natural lift.

Introduction

The study of Finsler spaces started in early 1980's by the paper works of R. Miron, I. Bucătaru, H. Shimada, S.V. Sabău. The main reason of this study is the phenomena of quantum gravity which is a part of theoretical physics with the main objective is to unify the quantum mechanics with general relativity where the study of small particles like photons is interesting when we change the notion of velocity into momenta. This is the fundamental reason why we started to study the geometry of the cotangent bundle. Another physical motivation of this study is that the pseudo-Finsler geometry is the most general geometry that involves a well defined notion of the arc length of a curve. The Finsler geometry provides the lagrangian formalism where the coordinates depend of position of the point on the manifold and the velocity through that point. The dual Finsler geometry instead provides the hamiltonian formalism where the coordinates depend of position of the point on the manifold and the momenta through that point.

Preliminaries

(I. Bucătaru) Let M be a smooth manifold of class C^∞, $dimM=n\in N^*$, arbitrary. We introduce the following notations:

- T_xM the tangent space at the point x to manifold M;
- (TM, π, M) the tangent bundle to manifold M, where $TM=\cup_{(x,\dot{x})}T_xM$;
- $\pi : TM \to M$ projection on the first coordinate, $\pi(x, \dot{x}) = x$.

The local coordinates $(x^i, \dot{x}^i)$ on the tangent bundle TM are naturally induced by local coordinates on M. We set $(x, \dot{x})$. TM is a vector space of dimension $2n$. The double tangent bundle is defined in (I. Bucătaru):

$$U_{(x,\dot{x})}T_{(x,\dot{x})}TM = TTM$$

Then, for every $(x, \dot{x})$ exists a natural local basis on $T_{(x,\dot{x})}TM$ such that:

$$B_{(x,\dot{x})} = \{ \partial_i = \frac{\partial}{\partial x^i}, \dot{\partial}_i = \frac{\partial}{\partial \dot{x}^i} \}$$

We denote the vertical subbundle with $VTM=\cup_{(x,\dot{x})\in TM}V_{(x,\dot{x})}TM$ the disjoint union of vertical subspaces which respects the condition:

$$VTM = Ker\{d\pi\},$$

where dπ is the linear approximation of the projection π. A local basis for VTM is $\{\dot{\partial}_i\}$. VTM is a natural construction, since the transformation matrix for the coordinates is the Jacobian.

(I. Bucătaru) **Definition 1:**

A nonlinear connection (Ehresmann connection) on TM is defined by a horizontal subbundle HTM which is supplementary to the vertical subbundle VTM:

$$TTM = HTM \oplus VTM$$

For constructing the horizontal subbundle is neccesary to define an Ehresmann connection because the change of basis on the horizontal subbundle depends on the coefficients of the Ehresmann connection and the basis of vertical subbundle. We denote with $HTM = \cup_{(x,\dot{x})\in TM}H_{(x,\dot{x})}TM$ the disjoint union of horizontal subspaces.

(I. Bucătaru) **Definition 2 :**

A smooth curve $c : t \in I \subset R \rightarrow c(t)=(x^i(t)) \in M$ is called autoparalled curve of nonlinear connection N if its extension (natural lift) to the tangent bundle TM is a horizontal curve. A curve is a horizontal curve if the tangent vector field is horizontal. The extension of the curve c on the tangent bundle TM is a curve $C : t \rightarrow (x^i(t),\dot{x}^i(t))$. The curve C has information about the velocity through the point x on the manifold M. In coordinates, the equation of the autoparallel curves is (I. Bucătaru):

$$\frac{d^2x^i}{dt^2} + N_j^i(x,\dot{x})\frac{dx^j}{dt} = 0,$$

where N^i_j are the coefficients of the nonlinear connection.

(R. Miron) **Definition 3 :**

A pseudo-Finsler space is a pair (M,L) where $L : A \rightarrow R,\ A \subset TM$ is a smooth function with the properties:

1. $L(x,\lambda\dot{x}) = \lambda^2 L(x,\dot{x})$, $\forall\lambda>0$;
2. $\forall(x,\dot{x}) \in A$ in any local chart the hessian matrix:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial\dot{x}^i\partial\dot{x}^j}$$

is nondegenerate.

Let $c : t \in [0,1] \rightarrow c(t) = (x^i(t)) \in M$. The extension C of the curve c on $TM\backslash\{0\}$ is defined by the equations:

$$x^i = x^i(t)\ ,\ \dot{x}^i = \frac{dx^i}{dt}$$

The natural (canonical) parameter s (R. Miron) :

$$s(t) = \int_{t0}^{t}\sqrt{L(x,\dot{x})}dt$$

Geodesics of the pseudo-Finsler space (M,L) are the extremals of L (R. Miron):

$$\frac{d^2x^i}{ds^2} + N_j^i\left(x,\frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

The adapted basis of the canonical nonlinear connection is:

$$\{ \delta_i = \partial_i - N_i^j \partial^\cdot_j \, , \partial^\cdot_i = \frac{\partial}{\partial \dot{x}^i} \}$$

The dual adapted basis (dual cobasis) is:

$$\{ dx^i \, , \delta\dot{x}^i = d\dot{x}^i + N_i^j dx^i \}$$

(I. Bucătaru) **Proposition 4**:

Dacă $c : [a , b] \to M$, $t \to (x^i(t))$ is a geodesic then it's extension $C : [a , b] \to TM$, $t \to (x^i(t) , \dot{x}^i(t))$ is a horizontal autoparallel curve if :

$$\delta\dot{x}^i \circ C = 0$$

If the equation $\delta_i L = 0$ is satisfied then the nonlinear connection is called metric connection.

Taking into consideration the Proposition 4, we can use another geometrical object to write the geodesic equations. The canonical spray (geodesic vector field) $S \in \chi(TM)$ is defined by:

$$S = \dot{x}^i \delta_i$$

Then $S = \dot{x}^i - 2G^i(x, \dot{x})$ where G^i represent the local coefficients of the canonical spray which are related to the coefficients of canonical nonlinear connection by formula:

$$N_i^j = \frac{\partial G^j}{\partial \dot{x}^i}$$

Using the local coefficients of the canonical spray now it is possible to write the geodesic equations on the pseudo-Finsler space in this form (I. Bucătaru):

$$\frac{d^2 x^i}{ds^2} + 2G^i(x, \dot{x}) = 0$$

(A. Bejancu) **Proposition 5:**

The extension C is the integral curves of the canonical spray S and they are the geodesics for the pseudo-Finsler space (M , L).

Acording to the paper work of (R. Miron) it is possible to associate the Berwald connection (which is linear) to the nonlinear connection.

(I. Bucătaru) **Definition 6**:

The Berwald connection associated to the nonlinear connection N is an operator D: $\chi(TM) \times \chi(TM) \to \chi(TM)$ with the following properties:

- $D_{\delta_i} \delta_j = G_{ji}^k \delta_k$
- $D_{\delta_i} \partial^\cdot_j = G_{ji}^k \delta_k$
- $D_{\partial^\cdot_i} \delta_j = 0$
- $D_{\partial^\cdot_i} \partial^\cdot_j = 0$,

Where $G_{ji}^k = \frac{\partial N_j^k}{\partial \dot{x}^i}$ are the coefficients of Berwald connection.

Making use of the Berwald connection now it is possible to introduce another important operator on the pseudo-Finsler space. The dynamical covariant derivative is an operator that measures the variation of the velocity through the geodesics of pseudo-Finsler space (M , L).

(I. Bucătaru) **Definition 7:**

The dynamical covariant derivative is the operator $\nabla : \chi^v \to \chi^v$ defined by:

$$\nabla Y = D_S Y \ , \ \forall \, Y \in \chi^v \, (TM)$$

Proposition 8:

The autoparallel curves equation can be write using the dynamical covariant derivative:

$$\nabla C^{\cdot} = 0$$

Main results

According to the studies of R. Miron, I. Bucătaru, H. Shimada, S.V. Sabău we will use the concept of Legendre duality presented in the paper work of (V.I.Arnold) between the lagrangian formalism and the hamiltonian formalism to introduce the dynamical covariant derivative operator on the dual Finsler space. In order to achieve the main results of this article, I will start this section with the following notations:

- $T_x{}^*M$ the cotangent space in point x to the manifold *M;*
- *(T^*M, π, M)* the cotangent bundle to the manifold *M,* where $T^*M = \cup_{x\in M} T_x{}^*M$.
- π is the projection on the first factor, $\pi(x,p) = x$.

The local coordinates (x^i, p_i) on T^*M are naturally induced by the coordinates on the base manifold *M.*

Definition 9:

A nonlinear connection (Ehresmann connection) N^* on T^*M is defined by a horizontal subbundle HT^*M of TT^*M supplementary to the vertical subbundle VT^*M which respects the Whitney sum decomposition:

$$TT^*M = HT^*M \oplus VT^*M,$$

where $HT^*M = Span\ \{\delta_i^*\}$ and $VT^*M = Span\ \{\partial^{\cdot i} := \frac{\partial}{\partial p_i}\}$.

The adapted basis of the nonlinear connection is:

$$\{\ \delta_i^* = \partial_i + N_{ji}^*(x,p)\partial^{\cdot j}, \partial^{\cdot i} = \frac{\partial}{\partial p_i}\}$$

The adapted dual basis (dual cobasis) is:

$$\{\ dx^i, \delta p_i = dp_i - N_{ji}^*(x,p)dx^j\},$$

where $N_{ji}^*(x,p)$ are the coefficients of the nonlinear connection.

The autoparallel curves equation of the nonlinear connection N^* are:

$$\frac{d^2x^i}{dt^2} + N_{ij}^*(x,p)\frac{dx^j}{dt} = 0$$

The Legendre transform $\mathfrak{L} : TM \rightarrow T^*M$ is a bijective application (because the hessian matrix of the lagrangian is nondegenerate) that, according to the paper work of (V.I.Arnold), connects the lagrangian and hamiltonian formalism through the formula:

$$H(x, p) = 2L(x, \dot{x}) - p_i\dot{x}^i$$

Thus, the geodesics of the dual Finsler space will be the geodesics of the pseudo-Finsler space *(M , L)* through the Legendre transform which are the solution curves of the Hamilton-Jacobi equations:

$$\frac{\partial H}{\partial p_i} = \dot{x}^i \quad , \quad \frac{\partial H}{\partial x^i} = -\frac{dp_i}{dt}$$

(D. Bao) **Definition 10:**

A dual Finsler space is a pair *(M , H)* where $H : A^* \rightarrow R$, $A^* \subset T^*M$ is a smooth map with the properties:

1. $H(x, \lambda p) = \lambda^2 H(x, p)$, $\forall \lambda > 0;$

2. $\forall (x, p) \in A$ in any local chart the hessian matrix:

$$g^{*ij} = \frac{1}{2}\frac{\partial^2 H}{dp_i dp_j}$$

is nondegenerate.

Definition 11:

The nonlinear connection N^* is a metric connection if the equation $\delta_i^* H = 0$ is satisfied.

Proposition 12:

The curve $c : [a,b] \to M,\ t \to (x^i(t))$ is a solution of the Hamilton-Jacobi equations if and only if it's extension $C^* : [a,b] \to TM,\ t \to (x^i(t), p_i(t))$ is the integral curve of the Hamiltonian vector field:

$$X^H = \frac{\partial H}{\partial p_i}\partial_i - \frac{\partial H}{\partial x^i}\partial^{\cdot i}$$

Proposition 13:

The Hamiltonian vector field has the following expression:

$$X^H = \frac{\partial H}{\partial p_i}\delta_i^* = \dot{x}^i \delta_i^*$$

Proof:

We use the expression of δ_i^* from the adapted basis.

After this step, we will substitute the partial derivatives of H with respect to x^i in the Hamiltonina vector field definition. Thus, we obtained the dual geometrical object X^H on T^*M which is the analogue geometrical object to the geodesic spray S on the tangent bundle TM.

Theorem 14:

The curve c is a geodesic of the dual space (M, H) if and only if the curve C^* satisfy either one of the equivalent conditions:

1. C^* is the integral curve of the Hamiltonian vector field X^H;
2. $dp_i \circ C^* = 0$;
3. C^* is an autoparallel curve (horizontal curve for the nonlinear connection N^*).

Proof:

Let $c : [a, b] \to M,\ c(t) = (x^i(t))$ be a geodesic on manifold M.

Consider the extension (natural lifted curve) $C^* : [a, b] \to T^*M,\ C^*(t) = (x^i(t), p_i(t))$.

We compute the tangent vectors along C^*:

$$C^*(t) = \dot{x}^i(t)\partial_i + \dot{p}_i(t)\partial^{\cdot i}$$

Now, we use the relation from the adapted basis:

$$\delta_i^* = \partial_i + N_{ji}^*(x,p)\partial^{\cdot j} \Rightarrow \partial_i = \delta_i^* - N_{ji}^*(x,p)\partial^{\cdot j}$$

The tangent vectors now along C^* have now the expression:

$$C^*(t) = \dot{x}^i(t)(\delta_i^* - N_{ji}^*(x,p)\partial^{\cdot j}) + \dot{p}_j(t)\partial^{\cdot j}$$

$$C^*(t) = \dot{x}^i(t)\delta_i^* + (\dot{p}_j(t) - \dot{x}^i(t)N_{ji}^*(x,p))\partial^{\cdot j}$$

Now, using the fact that c is a geodesic implies that C^* is a horizontal curve so C^* must be a horizontal curve. Thus, we obtain the following relation:

$$C^*(t) = \dot{x}^i(t)\delta_i^* = X^H$$

In conclusion, C^* is the integral curve of the Hamiltonian vector field X^H. This is also equivalent to the fact that:

$$\dot{p}_j(t) - \dot{x}^i(t)N_{ji}^*(x,p) = 0$$

which lead us to conclude that $dp_i \circ C^* = 0$ and C^* is an autoparallel curve.

Definition 15:

We define the Berwald connection of the nonlinear connection N^* as an operator $D^* : \chi (T^*M) \times \chi (T^*M) \rightarrow \chi (T^*M)$ which respect the following properties:

- $D^*{}_{\delta_i}\delta_j = G_{ji}^{*k}\delta_k$
- $D^*{}_{\delta_i}\partial^{\cdot}{}_j = G_{ji}^{*k}\delta_k$
- $D^*{}_{\partial^{\cdot}{}_i}\delta_j = 0$
- $D^*{}_{\partial^{\cdot}{}_i}\partial^{\cdot}{}_j = 0$,

Where $G_{ik}^{*j} = \frac{\partial N_{ik}^{*}}{\partial p_j}$ are coefficients of the Berwald connection associated to the nonlinear connection N^*.

We introduce the following notations:

χ^{*v} – the set of the vertical vector fields on T^*M;

χ^{*h} – the set of the horizontal vector fields on T^*M.

$$Y^V \in \chi^{*v} \Leftrightarrow Y^v = Y_i(x, p)\partial^{\cdot i}$$

$$Y^h \in \chi^{*h} \Leftrightarrow Y^h = Y_i(x, p)\delta^{\cdot i}$$

Finally, we can now introduce the dynamical covariant derivative operator ∇^* on the cotangent bundle T^*M. The definition of the dynamical covariant derivative depends on the Berwald connection associated to the Ehresmann connection N^* as will see in the following:

Definition 16:

We define the dynamical covariant derivative on T*M in the following way:

$$\nabla^* : \chi^{*V} \rightarrow \chi^{*V}, \ \nabla^* Y^v = D^*_{\chi^H} Y^v$$

$$\nabla^* : \chi^{*h} \rightarrow \chi^{*h}, \ \nabla^* Y^h = D^*_{\chi^H} Y^h$$

Theorem 17:

The autoparallel curves equation can be expressed using the dynamical covariant derivative operator ∇^* introduced in the previous definition:

$$\nabla^* C^{\cdot *} = 0$$

Proof:

We consider a geodesic c on the base manifold M and it's natural lift C^* to the cotangent bundle T^*M. Then, computing the tangent vectors along the curve C^*, using the relation between δ_i^* and ∂_i from the adapted basis and the fact that C^* is a horizontal curve for nonlinear connection N^*.

In conclusion, we use the definition of the dynamical covariant derivative operator ∇^* as a Berwald connection with respect to the Hamiltonian vector field and then we apply the Theorem 14.

Conclusions:

The results obtained in the previous section of this article shows that our theory developed on the cotangent bundle represents the dual hamiltonian formalism and respect the same properties as already known on the tangent bundle. With the results already obtained, we want to push forward our theory and finding the geodesic deviation equations and after that, by taking account to Pirani's principle to write the field equation on the cotangent bundle. In the next part, I want to show an example of constructing a tangent bundle of a one dimensional manifold (circle in our case) .

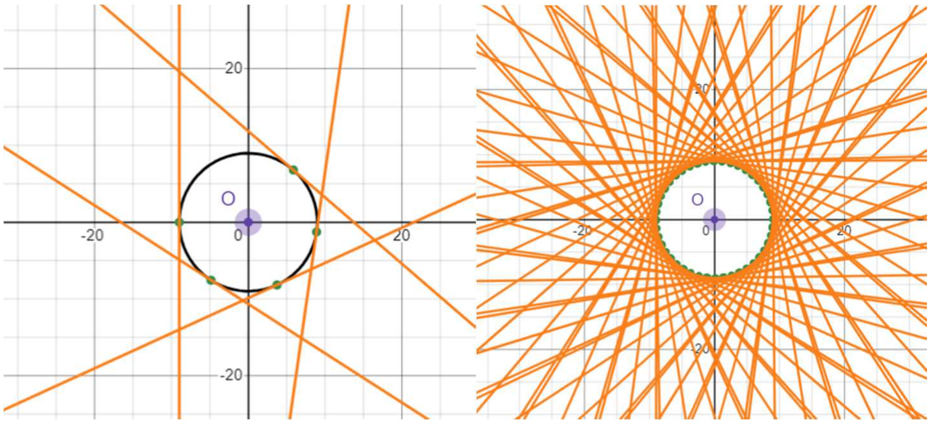

Fig. 1. An example of a tangent bundle construction of an one dimensional manifold. For these images I used the mathematical soft Desmos. Autor: Bucătaru Cosmin

Tangent Bundle construction animation link

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